Exact expression for the diffusion propagator in a family of time-dependent anharmonic potentials

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We have obtained the exact expression of the diffusion propagator in the time-dependent anharmonic potential $V(x,t)=\frac{1}{2}a(t)x^2+b\ln x$. The underlying Euclidean metric of the problem allows us to obtain analytical solutions for a whole family of the elastic parameter $a(t)$, exploiting the relation between the path integral representation of the short time propagator and the modified Bessel functions. We have also analyzed the conditions for the appearance of a nonzero flow of particles through the infinite barrier located at the origin $(b<0)$. [S1063-651X(99)14708-1]

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I. INTRODUCTION

The mathematical theory of stochastic processes has proven to be not only a useful but also a necessary tool when studying physical, chemical, and biological systems under the effect of fluctuations [1]. Recent theoretical and experimental studies have shown that there are many situations where fluctuations play an essential role leading to new phenomena induced by the presence of noise. A few examples of such situations are some problems related with self-organization and dissipative structures [2,3], noise-induced transitions [4], noise-induced *phase* transitions [5], noise sustained patterns [6], stochastic resonance in zero-dimensional and in spatially extended systems [7,8].

An almost natural way to describe (Markovian) stochastic processes corresponds to an approach introduced by Wiener based on a sum over trajectories [9], anticipating by two decades Feynman's work on path integrals [10]. This approach was later applied by Onsager and Machlup to some Markovian nonequilibrium processes [11]. However, even some non-Markov processes can be also described within this framework [12]. In spite of the historical fact, path and functional integration methods have been largely studied and applied within the quantum mechanical realm, while its study and application in stochastic processes is scarce [13]. However, it is worth here to remark that, following earlier results [14], path and functional integral methods have been widely applied in the theory of dynamic critical phenomena, both at equilibrium and nonequilibrium phase transitions [15].

Among others, one of the problems that has not received much attention within the path integral description of stochastic processes is the application of space-time transformations, while this kind of transformations have been largely used within the realm of quantum mechanics [16]. Among the few studies in this regard (see Ref. [17]) a recent one refers to a transformation relating the diffusion propagator in a time-dependent *harmonic* oscillator with the propagator for the case of free diffusion, including a whole family of possible analytical solutions [18]. Also, a formal adaptation of Duru-Kleinert-like transformations to the stochastic case, overcoming the main disadvantage of the direct application of such transformations, namely, that Duru-Kleinert transformations do not link different Markov processes, was introduced in Ref. [19]. However, it is worth mentioning the treatment of transformations between ''different'' Wiener processes done in Friendlin's book [indicated with (j) in Ref. [13]], even though they are not the exact equivalent to the Duru-Kleinert transformations.

In this paper we present the exact solution for the diffusion propagator in a time-dependent anharmonic oscillator $V(x,t)=\frac{1}{2}a(t)x^2+b\ln x$. This particular choice of the potential can be useful to model the behavior of several physical and biological systems. Among them, the study of neuron models (e.g., integrate and fire models [7]), stochastic resonance in monostable nonlinear oscillators [20] and its possible application to spatially extended systems [8]. Also, we can consider that the logarithmic term, within the path integral scheme (or as a Boltzmann-like weight) mimics a prefactor corresponding to an effective energy barrier. It is clear that the possibility of having exact expressions of the stochastic propagator in a nonsymmetrical potential can be of interest. In fact, in many of the above mentioned applications (specially on neuron models) the potential studied in this work would represent a more realistic approximation to the real behavior of the system under study. In other problems such us Brownian motors, this asymmetry is not just an improvement but an unavoidable ingredient of the model.

The approach used in this work has been inspired by the

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solution presented in Ref. [21], corresponding to the exact quantum mechanical propagator in a time-dependent harmonic potential plus a singular perturbation. In the present case, the fact that the metric of the underlying space is Euclidean allows us to obtain the exact analytical expression of the diffusion propagator for a whole family of functional forms of the time-dependent elastic parameter.

In the next section we introduce the model we are going to study and show the procedure to be followed in order to obtain the exact form of the propagator. We also discuss the presence of noise-induced flow of particles through the infinite barrier located at the origin, provided that the noise amplitude is large enough for the particles to overcome the deterministic drift. In Sec. III we show how to obtain a family of analytical solutions. In the last section we make a final discussion and comment on the possible applications of the present results.

II. DIFFUSION PROPAGATOR

In this section we will follow, and adequately adapt, the results of the paper of Khandekar and Lawande [21]. Our starting point is to consider the following Langevin equation:

$$\dot{x}=h(x,t)+\xi(t), \tag{1}$$

where $\xi(t)$ is an additive *Gaussian white noise* [1]. That is it fulfills the conditions $\langle\xi(t)\rangle=0$ and $\langle\xi(t)\xi(t')\rangle=2D\delta(t-t')$. This equation describes the overdamped motion of a particle in a time-dependent potential. In this work, we will consider the force term $h(x,t)=-a(t)x-b/x$ which, through the relation $h(x,t)=-\partial V/\partial x$, corresponds to the following potential:

$$V(x,t)=\frac{1}{2}a(t)x^2+b\ln x. \tag{2}$$

This potential is defined for $x>0$ and, whenever $b<0$ and the elastic parameter $a(t)$ is positive, it corresponds to an anharmonic monostable system composed by a time-dependent harmonic oscillator plus a logarithmic term which is singular at the origin.

As it will be shown later, even in this monostable situation, the noise could be able to induce a flow of particles through the infinite barrier located at the origin, overcoming the deterministic drift whenever $D>-2b$ holds. In fact, the meaningful condition related to the conservation of particles inside the system (zero flux at $x=0$) is $D<-2b$.

In this work we will relax the monostability condition, allowing for the time-dependent elastic parameter to take negative values. We will show that in this extended situation an asymptotic probability distribution can be reached whenever the elastic term satisfies

$$\lim_{t_0\to-\infty}\int_{t_0}^{t}a(s)ds=\infty \quad \forall \ t. \tag{3}$$

We will say that in this case the potential is *strongly attractive*.

The path integral representation of $P(x_b,t_b|x_a,t_a)$, that is the transition probability associated with this Langevin equation, is given by (see, for instance, [13i])

$$P(x_b,t_b|x_a,t_a)=\int_{x(t_a)=x_a}^{x(t_b)=x_b}\mathcal{D}[x(t)]\times\exp\left[-\int_{t_a}^{t_b}L[x(\tau),\dot{x}(\tau),\tau]d\tau\right]. \tag{4}$$

Here the stochastic *Lagrangian* or *Onsager-Machlup* [11] functional is given, in a midpoint discretization, by

$$L(x,\dot{x},t)=\frac{1}{2D}[\dot{x}-h(x,t)]^2+\frac{1}{2}\frac{\partial h(x,t)}{\partial x}. \tag{5}$$

Replacing the actual form of $h(x,t)$ the previous expression can be expanded to yield

$$L=L_0+\frac{d\Phi}{dt}, \tag{6}$$

where Φ corresponds to

$$\Phi(t)=\left[\frac{b}{D}-\frac{1}{2}\right]\int_{t_0}^{t}a(\tau)d\tau+\frac{b}{D}\ln x+\frac{a(t)x^2}{2D}, \tag{7}$$

with arbitrary t_0, and

$$L_0\equiv\frac{1}{2D}\left[\dot{x}^2+[a(t)^2-\dot{a}(t)]x^2+(b+D)\frac{b}{x^2}\right]. \tag{8}$$

Hence the path integral in Eq. (4) adopts the form

$$P(x_b,t_b|x_a,t_a)=e^{-[\Phi(t_b)-\Phi(t_a)]}K(x_b,t_b|x_a,t_a), \tag{9}$$

with

$$K(x_b,t_b|x_a,t_a)=\int_{x_a}^{x_b}\mathcal{D}[x(t)]\times\exp\left[-\frac{1}{2D}\int_{t_a}^{t_b}\left(\dot{x}^2+\omega(\tau)x^2+(b+D)\frac{b}{x^2}\right)d\tau\right], \tag{10}$$

and $\omega(t)=a(t)^2-\dot{a}(t)$. As usual, the path integral in Eq. (10) is defined in a discretized form by

$$K(x_b,t_b|x_a,t_a)=\lim_{N\to\infty}A_N\int\cdots\int\times\exp\left(-\sum_{j=1}^{N}S_j(x_j,x_{j-1})\right)\prod_{j=1}^{N-1}dx_j, \tag{11}$$

with $N\varepsilon=t_a-t_b$, $t_j=t_a+j\varepsilon$, $x_0=x_a$; $x_N=x_b$; $A_N=[2\pi D\varepsilon]^{-N/2}$; and

$$S_j=S_j(x_j,x_{j-1})=\varepsilon L_o(x_j,x_{j-1})$$
$$=\frac{1}{2D}\left[\frac{x_j^2+x_{j-1}^2}{\varepsilon}+\varepsilon\omega_j x^2\right]-\left[\frac{x_jx_{j-1}}{D\varepsilon}+\frac{D\left(\theta^2-\frac{1}{4}\right)\varepsilon}{2x_jx_{j-1}}\right].$$

Here

$$\theta=\theta(b)=\frac{1}{2}\sqrt{1+\frac{4b(b+D)}{D^2}}=\left|\frac{1}{2}+\frac{b}{D}\right|. \qquad (12)$$

Up to first order in ε (exploiting that $\varepsilon \ll 1$) we can use the following asymptotic form of the modified Bessel function:

$$\exp\left[\frac{u}{\varepsilon}-\frac{1}{2}\left(\theta(b)^2-\frac{1}{4}\right)\frac{\varepsilon}{u}+O(\varepsilon^2)\right]\approx\sqrt{\frac{2\pi u}{\varepsilon}}I_{\theta(b)}\left(\frac{u}{\varepsilon}\right). \qquad (13)$$

Using the last expression with $u=x_j x_{j-1}/D$ the propagator of Eq. (11) may be cast to the following form:

$$K(x_b,t_b|x_a,t_a)=\lim_{N\to\infty}\left(\frac{1}{2\pi D\varepsilon}\right)^{1/2}\int\dots\int\prod_{j=1}^{N-1}dx_j\prod_{j=1}^{N}\times\exp\left[-\frac{1}{2D\varepsilon}(x_j^2+x_{j-1}^2+\varepsilon^2\omega_j x_j^2)\right]\times\left(\frac{2\pi x_j x_{j-1}}{D\varepsilon}\right)^{1/2}I_\theta\left(\frac{x_j x_{j-1}}{D\varepsilon}\right). \qquad (14)$$

The last expression can be rewritten as

$$K(x_b,t_b|x_a,t_a)=\exp\left[\frac{-\beta}{2}(x_a^2+x_b^2)\right]\lim_{N\to\infty}\beta^N\int\dots\int\prod_{j=1}^{N-1}\times e^{-\alpha_j x_j^2}I_\theta(\beta x_j x_{j-1})x_j dx_j, \qquad (15)$$

where

$$\alpha_j=\beta\left(1+\frac{\varepsilon^2}{2}\omega_j\right),\quad 0\le j\le N-1,\quad \beta=\frac{1}{D\varepsilon}.$$

Now, in order to perform the integrations of Eq. (15), we can use the equality known as Weber formula [22,23], which is given by

$$\int_0^\infty e^{-\alpha x^2}I_\theta(ax)\,I_\theta(bx)x\,dx=\frac{1}{2\alpha}\exp\left[\frac{a^2+b^2}{4\alpha}\right]I_\theta\left(\frac{ab}{2\alpha}\right),$$

and is valid for Re(θ)>-1, Re(α)>0 (here, both conditions are fulfilled). The final result is

$$K(x_b,t_b|x_a,t_a)=\sqrt{x_a x_b}\lim_{N\to\infty}a_N\,e^{(p_N x_a^2+q_N x_b^2)}I_\theta(a_N x_a x_b), \qquad (16)$$

where the quantities a_N, p_N, and q_N are defined in Appendix A. These quantities are related to a function $Q(t)$ that obeys the equation (as usual, we have indicated time derivatives with dots)

$$\ddot{Q}(t)-\omega(t)Q(t)=0, \qquad (17)$$

with the initial condition $Q_0=Q(t_a)=0$. In the limit $\varepsilon\to 0 (N\to\infty)$, we find that (see Appendix A)

$$\lim_{N\to\infty}a_N=\frac{1}{D}\frac{\dot{Q}(t_a)}{Q(t_b)}, \qquad (18)$$

$$\lim_{N\to\infty}p_N=\lim_{\varepsilon\to 0}\left(\frac{1}{\varepsilon}-\dot{Q}^2(t_a)\int_{t_a+\varepsilon}^{t_b}\frac{dt}{Q(t)^2}\right), \qquad (19)$$

$$\lim_{N\to\infty}q_N=-\frac{1}{2D}\frac{\dot{Q}(t_b)}{Q(t_b)}. \qquad (20)$$

To calculate the second limit, it is necessary to solve Eq. (17). As it was shown in Ref. [18], the complete solution of Eq. (17) can be reduced to *quadratures*, with the general form given by

$$Q(t)=k_1\,\mathcal{R}_{t_a}(t)+k_2\,\mathcal{S}_{t_a}(t), \qquad (21)$$

where

$$\mathcal{R}_{t_a}(t)=\exp\left(-\int_{t_a}^{t}a(s)ds\right)$$

$$\mathcal{S}_{t_a}(t)=\exp\left(-\int_{t_a}^{t}a(s)ds\right)\int_{t_a}^{t}\exp\left(2\int_{t_a}^{\tau}a(s)ds\right)d\tau. \qquad (22)$$

Hence, the solution fulfilling the initial condition $Q_0=Q(t_a)=0$ is

$$Q(t)=\dot{Q}(t_a)\mathcal{S}_{t_a}(t). \qquad (23)$$

After replacing this solution into the expressions for a_N, p_N and q_N (see appendix B), we finally arrive to a completely analytical expression for the transition probability

$$P(x_b,t_b|x_a,t_a)=e^{-[\Phi(t_b)-\Phi(t_a)]}\frac{\sqrt{x_a x_b}}{D\,\mathcal{S}_{t_a}(t_b)}I_\theta\left(\frac{x_a x_b}{D\,\mathcal{S}_{t_a}(t_b)}\right)\times\exp\left(\frac{-1}{2D\,\mathcal{S}_{t_a}(t_b)}\left[[\mathcal{R}_{t_a}(t_b)+a(t_a)]x_a^2+\left(\frac{1}{\mathcal{R}_{t_a}(t_b)}-a(t_b)\right)x_b^2\right]\right), \qquad (24)$$

which can be further simplified as

$$P(x_b,t_b|x_a,t_a)=x_a^{b/D+1/2}x_b^{-(b/D)+1/2}\frac{[\mathcal{R}_{t_a}(t_b)]^{b/D-1/2}}{D\,\mathcal{S}_{t_a}(t_b)}I_\theta\left(\frac{x_a x_b}{D\,\mathcal{S}_{t_a}(t_b)}\right)\times\exp\left(\frac{-1}{2D\,\mathcal{S}_{t_a}(t_b)}\left[\mathcal{R}_{t_a}(t_b)x_a^2+\frac{1}{\mathcal{R}_{t_a}(t_b)}x_b^2\right]\right). \qquad (25)$$

It is straightforward to check for some particular choices of $a(t)$ and b that the last expression fulfills the corresponding Fokker-Planck equation. Albeit not so simple, we have also proved it for the general case. The last expression also indicates that, in order to have the explicit form of the propagator we only need to obtain the function $Q(t)$ [the solution of Eq. (17) given in Eq. (21)] for the problem under study [that is, for a given form of the function $a(t)$]. We will provide a family of solutions for a rather general form of the function $a(t)$ in a subsequent section. Before that, we will discuss the possibility of finding a net current at the origin.

A. Flow through the infinite barrier

Let us first evaluate from Eq. (25) the asymptotic probability distribution, that is

$$P(x,t)=\lim_{t_a\to-\infty} P(x,t|x_a,t_a). \tag{26}$$

In the strongly attractive case [Eq. (3)], it can be easily shown that $\mathcal{S}_{t_a}(t)$ diverges and that $\mathcal{R}_{t_a}(t)$ goes to zero as $t_a\to-\infty$. Thus, we will make use of the expansion of the modified Bessel function for small argument [22],

$$I_\theta(z)=\frac{1}{\Gamma(\theta+1)}\left(\frac{z}{2}\right)^\theta+O(z^{\theta+2}). \tag{27}$$

Replacing this expansion into Eq. (25), we get

$$P(x,t)=\left[\lim_{t_a\to-\infty}[x_a\,\mathcal{R}_{t_a}(t)]^{b/D+1/2+\theta}\right]\times\frac{2}{\Gamma(\theta+1)}\,\frac{x^{-(b/D)+1/2+\theta}}{[2\,D\,g(t)]^{1+\theta}}\exp\left(-\frac{x^2}{2\,D\,g(t)}\right), \tag{28}$$

where $g(t)$ is defined as

$$g(t)=\lim_{t_a\to-\infty}\mathcal{S}_{t_a}(t)\mathcal{R}_{t_a}(t). \tag{29}$$

It is clear that unless the condition

$$1/2+b/D+\theta=0 \tag{30}$$

holds, the system cannot reach an asymptotic probability distribution. In fact, the term between the square brackets in Eq. (28) depends on the initial condition. Furthermore, it can be shown that the normalization of Eq. (28) gives a vanishing function of t unless the previous condition holds. Note that Eq. (30) implies, through Eq. (12), $D<-2b$. This condition gives the maximum value of noise amplitude for the particles to be confined inside the interval $(0,\infty]$.

This encourages us to show explicitly the existence of a noise-induced probability current through the infinite barrier at the origin when $D>-2b$. Before giving a rigorous deduction, let us state a simple argument which provides some clue about the underlying physical mechanism governing this flow. As it is clear from the Langevin equation, the particle is subjected to both deterministic and stochastic forces. If we analyze separately both contributions to the particle movement near the origin, we obtain for the deterministic trajectory $x_d(t)=\sqrt{-2bt}$. Comparing this result with the well known diffusive behavior, where the uncertainty on the particle's position grows as $x_s=\sqrt{Dt}$, we reobtain the previous condition $D>-2b$ for the possible appearance of noise-induced leakage of particles.

The probability current at the origin $J(x=0,t|x_a,t_a)$ can be evaluated from the associated Fokker-Planck equation. In the case $D>-2b$ we obtain

$$J(x=0,t|x_a,t_a)=\left(-ax-\frac{b}{x}-\frac{D}{2}\frac{\partial}{\partial x}\right)P(x,t|x_a,t_a)|_{x=0}$$
$$=-D\left(\frac{b}{D}+\frac{1}{2}\right)J_{x_a,t_a}(t), \tag{31}$$

where it can be shown that $J_{x_a,t_a}(t)$ is a positive function of time for any given initial condition. Therefore, we have obtained a nonzero negative current, as previously stated. (This somewhat counterintuitive flux has an interesting quantum counterpart in the ''fall to the center'' effect studied by Landau [24].)

B. Asymptotic probability distribution

In the case $D<-2b$ there is no probability leakage. In fact, the asymptotic probability distribution can be obtained from Eq. (32) and is given by

$$P(x,t)=\frac{2}{\Gamma(\theta+1)}\,\frac{x^{-(2b/D)}}{[2\,D\,g(t)]^{1+\theta}}\exp\left(-\frac{x^2}{2\,D\,g(t)}\right), \tag{32}$$

which can be easily shown to be normalizable.

It is worth studying how the properties of the elastic parameter function $a(t)$ influences the behavior of the function $g(t)$, which reflects the time evolution of the width of the probability distribution. First, note that from the definition of $g(t)$ given in Eq. (29) we can obtain

$$\dot{g}(t)=-2a(t)g(t)+1. \tag{33}$$

From this equation it can be deduced that $g(t)>0\ \forall\ t$, as must be expected for any well behaved probability distribution. In addition, it can be proved that in order to confine the particle in a small region of width $\sqrt{g(t)}\sim\sqrt{\epsilon}$ an attractive force of order $a(t)\sim1/\epsilon$ is needed. On the other hand, a small attractive force of order $a(t)\sim\epsilon$, gives a broad distribution with $g(t)\sim1/\epsilon$. The limiting cases for $P(x,t)$ corresponding to an unbounded spreading $[a(t)\to0\Rightarrow g(t)\to\infty]$, and to an asymptotically approach to a $\delta(x)$ distribution $[a(t)\to\infty\Rightarrow g(t)\to0]$, can be also obtained.

From the previous paragraph it is clear that even in the strongly attractive situation [see Eq. (3)] the probability distribution may exhibit an unbounded spreading. In fact, we have already shown that even in the monostable situation $a(t)>0$, but where the strength vanishes in time, $g(t)$ grows indefinitely. Therefore, in order to obtain a nondivergent width of the probability distribution, the conditions on the attractive term have to be stronger than the one imposed by Eq. (3). We may infer that the localized-probability condition should be related to a non-vanishing attractive strength of the time averaged potential.

C. General localization conditions

Let us discuss the set of conditions that ensures the asymptotic localization of the probability distribution. From the analysis of Eq. (33) it is clear that in order to guarantee a nondivergent $g(t)$ the elastic parameter should have the following properties. Its accumulated strength is positive, i.e.,

$$\int_{t_i}^{t} a(\tau)\, d\tau = c > 0 \quad \forall t \tag{34}$$

where c is an arbitrary constant and t_i is the nearest time which fulfills the previous equation. This condition is clearly fulfilled if the potential is strongly attractive. We may also infer that the accumulated attractive effect [Eq. (34)] should be nonvanishing. In other words, the elapsed time where the accumulated strength reaches the given constant c is bounded, that is

$$t - t_i = \Delta t \leqslant \Delta t_u \quad \forall t \tag{35}$$

where $\Delta t_u \equiv \Delta t_u(c)$ is the mentioned upper bound for the elapsed time. It is evident that condition (35) is more restrictive than the one imposed by Eq. (3).

In the following section we will provide a family of examples where the probability is asymptotically localized.

III. FAMILY OF ANALYTICAL SOLUTIONS

As already mentioned, to obtain the final expression for the diffusion propagator, we must first solve Eq. (17) for a given choice of the function $a(t)$. Because the frequency $\omega(t)$ depends only on the harmonic term of the potential, we can make use of any known solution of the simpler harmonic case. A method to generate a whole family of analytical solutions has been proposed in Ref. [18] for the time-dependent harmonic oscillator. In order to reach such a goal the elastic parameter was written in the following form:

$$a(t) = f(t) + \frac{1}{2}\frac{\dot{f}(t)}{f(t)}. \tag{36}$$

This allows us to find the corresponding independent solutions of Eq. (17),

$$q_1(t) = \frac{\sinh[F(t)]}{\sqrt{f(t)}}, \tag{37}$$

$$q_2(t) = \frac{\cosh[F(t)]}{\sqrt{f(t)}}, \tag{38}$$

where $F(t) = \int_{t_0}^{t} f(s)\, ds$, indicating that $f(t)$ must be an integrable function. The solution that satisfies the initial condition $Q(t_a) = 0$ reduces to

$$Q(t) = \dot{Q}(t_a)\frac{\sinh[F(t) - F(t_a)]}{\sqrt{f(t) f(t_a)}}. \tag{39}$$

With this result, the transition probability in Eq. (25) adopts the analytical form

$$P(x_b, t_b | x_a, t_a) = \mathrm{e}^{[1/2 - b/D][F(t_b) - F(t_a)]}\left(\frac{f(t_a)}{f(t_b)}\right)^{b/2D - 1/4} x_a^{b/D + 1/2} x_b^{-(b/D) + 1/2} \frac{\sqrt{f(t_b) f(t_a)}}{D \sinh[F(t_b) - F(t_a)]}$$
$$\times I_{\theta(b)}\left(\frac{x_a x_b}{D}\frac{\sqrt{f(t_b) f(t_a)}}{\sinh[F(t_b) - F(t_a)]}\right)\exp\left(-\frac{f(t_a) e^{-[F(t_b) - F(t_a)]} x_a^2 + f(t_b) e^{[F(t_b) - F(t_a)]} x_b^2}{2D \sinh[F(t_b) - F(t_a)]}\right). \tag{40}$$

Hence, we have obtained a completely analytical expression for the propagator in Eq. (40), which only depends on the choice of the elastic parameter $a(t)$.

IV. FINAL REMARKS

In this work we obtained the exact expression for the diffusion propagator in the time-dependent anharmonic potential $V(x,t) = \frac{1}{2}a(t)x^2 + b \ln x$ for a rather general choice of the elastic parameter. The knowledge of the exact form of the propagator can be useful to model different physical and biological phenomena. Particularly interesting problems, suitable to be studied taking advantage of this results, are realistic nonsymmetric neuron membrane potentials [25] and the phenomenon of stochastic resonance in a monostable zero-dimensional potential, in spatially extended systems, and in several neuron firing models, among others. A complete and recent review of these stochastic resonance topics can be found in the work of Gammaitoni *et al.* [7]. On the other hand, the knowledge of the exact propagator in the indicated time-dependent anharmonic potential can be useful as a benchmark to test approximate numerical or analytical procedures. Among them we can refer to some of the problems discussed in [26].

Among the several studies of stochastic resonance in monostable systems, it has been shown using scaling arguments and numerical experiments, that the signal to noise ratio is a monotonically increasing function of the noise amplitude [27]. By contrast, it is quite clear that this increase in the response of the system cannot be unbounded. We shall present elsewhere our own results concerning the phenomenon of stochastic resonance in a system described by the potential discussed in the present paper. In this regard, the relation between the maximum noise amplitude and the deterministic force near the origin proved to be a meaningful cutoff for the increase of the response.

It is worth remarking here that the limit $b \to 0$ is a (kind of) singular one. The naive point of view will be that, in such a limit, the form of the propagator in Eq. (25) shall reduce to the one corresponding to the case of the harmonic time-dependent potential $V(x) \sim \frac{1}{2}a(t)x^2$. However, this limit corresponds to a harmonic time-dependent potential for $x > 0$

with an absorbing boundary condition at $x=0$. Then, it should be possible to reobtain the limit $b\to 0$ of the diffusion propagator found in this paper $[P_0(x_b,t_b|x_a,t_a)]$ from the one obtained in Ref. [18] for the harmonic time-dependent case $[P_h(x_b,t_b|x_a,t_a)]$ simply as $P_0(x_b,t_b|x_a,t_a) = P_h(x_b,t_b|x_a,t_a) - P_h(-x_b,t_b|x_a,t_a)$. It can be easily proved that this is indeed the case.

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APPENDIX A

The quantities a_N, p_N, and q_N are defined according to

$$a_N = \beta \prod_{j=1}^{N-1} \frac{\beta}{2\gamma_j}; \quad \gamma_1 = \alpha_1, \quad \gamma_j = \alpha_j - \frac{\beta^2}{4\gamma_{j-1}};$$

$$p_N = -\frac{\beta}{2} + \sum_{j=1}^{N-1} \frac{\beta_j^2}{4\gamma_j}; \quad q_N = -\frac{\beta}{2} + \frac{\beta^2}{4\gamma_{N-1}}; \tag{A1}$$

$$\beta_1 = \beta; \quad \beta_j = \beta \prod_{k=1}^{j-1} \frac{\beta}{2\gamma_k}.$$

In order to determine the limiting (when $N\to\infty$) values of a_N, p_N, and q_N, it is useful to define the following auxiliary quantities:

$$\lambda_j = \frac{2}{\beta}\gamma_j, \quad \Lambda_k = \prod_{j=1}^{k} \frac{1}{\lambda_j}, \tag{A2}$$

with α_j as defined after Eq. (14) and γ_j adopting the form

$$\gamma_j = \beta\left(1 + \frac{\varepsilon^2\omega_j}{2}\right) - \frac{\beta^2}{4\gamma_{j-1}}, \tag{A3}$$

which allows us to obtain the following equation for λ_j:

$$\lambda_j = 2\left(1 + \frac{\varepsilon^2\omega_j}{2}\right) - \frac{1}{\lambda_{j-1}}. \tag{A4}$$

If we now define that $\lambda_j = Q_{j+1}/Q_j$, the last equation can be rewritten as

$$Q_{j+1} - 2Q_j + Q_{j-1} = \omega_j \varepsilon^2 Q_j, \tag{A5}$$

which, in the limit $N\to\infty$ (and $\varepsilon\to 0$), becomes Eq. (17), with the initial condition $Q_0 = Q(t_a) = 0$, which follows from Eq. (A5).

Finally, we can express the coefficients a_N, p_N, and q_N as functions of the new variables

$$a_N = \beta\Lambda_{N-1} = \beta\frac{Q_1}{Q_N} = \beta\frac{Q_1 - Q_0}{Q_N} = \beta\varepsilon\left(\frac{Q_1 - Q_0}{\varepsilon}\right)\frac{1}{Q_N}, \tag{A6}$$

$$p_N = -\frac{\beta}{2}\left(1 - \sum_{j=1}^{N-1} \frac{Q_1^2}{Q_{j+1}Q_j}\right) \tag{A7}$$

$$q_N = -\frac{\beta\varepsilon}{2}\left(\frac{Q_N - Q_{N-1}}{\varepsilon Q_N}\right). \tag{A8}$$

APPENDIX B

The replacement of the general solution for $Q(t)$ indicated in Eq. (23) into Eqs. (18)–(20), leads us to obtain the limiting values of a_N, p_N, and q_N. For p_N we find

$$\lim_{N\to\infty} p_N = \lim_{\varepsilon\to 0} \frac{-1}{2D}\left(\frac{1}{\varepsilon} + \frac{1}{\int_{t_a}^{t_b} d\tau \exp\left(2\int_{t_a}^{\tau} a(\varsigma)d\varsigma\right)} - \frac{1}{\int_{t_a}^{t_a+\varepsilon} d\tau \exp\left(2\int_{t_a}^{\tau} a(\varsigma)d\varsigma\right)}\right). \tag{B1}$$

Making a Taylor expansion up to second order in ε of the last denominator we can calculate the limit in Eq. (B1) yielding

$$\lim_{N\to\infty} p_N = \frac{-1}{2D}\left(\frac{1}{\int_{t_a}^{t_b} d\tau \exp\left(2\int_{t_a}^{\tau} a(\varsigma)d\varsigma\right)} + a(t_a)\right). \tag{B2}$$

The expressions for a_N and q_N, in terms of the explicit form for $Q(t)$ results in

$$\lim_{N\to\infty} q_N = \frac{-1}{2D}\left(\frac{\exp\left(\int_{t_a}^{t_b} a(\varsigma)d\varsigma\right)}{\int_{t_a}^{t_b} d\tau \exp(2\int_{t_a}^{\tau} a(\varsigma)d\varsigma)} - a(t_b)\right), \tag{B3}$$

$$\lim_{N\to\infty} a_N = \frac{1}{D}\frac{\exp\left(\int_{t_a}^{t_b} a(\varsigma)d\varsigma\right)}{\int_{t_a}^{t_b} d\tau \exp\left(2\int_{t_a}^{\tau} a(\varsigma)d\varsigma\right)}. \tag{B4}$$

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